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$\mathfrak{h}$-NORMALIZERS AND LOCAL DEFINITIONS OF SATURATED FORMATIONS OF FINITE GROUPS

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ABSTRACT

We define, in each finite group G, $\mathfrak{h}$-normalizers associated with a Schunck class $\mathfrak{h}$ of the form $E_{\Phi}\mathfrak{f}$ with $\mathfrak{f}$ a formation. We use these normalizers in order to give some sufficient conditions for a saturated formation of finite groups to have a maximal local definition.

1. Introduction

The celebrated Carter–Hawkes $\mathfrak{f}$-normalizers of a soluble group have been a source of inspiration of numerous works always in the soluble (or at most π-soluble) universe. In this paper we introduce the $\mathfrak{h}$-normalizers of a finite, non-necessarily soluble, group where $\mathfrak{h}$ is a Schunck class of the form $E_{\Phi}\mathfrak{f}$ with $\mathfrak{f}$ a formation and give some applications on normal complementation and local definitions of saturated formations of finite groups. We prove that in the theory of $\mathfrak{f}$-normalizers, the solubility hypothesis can be weakened to the solubility of the $\mathfrak{f}$-residual to obtain the main results of the classical theory: conjugation, cover and avoidance property and relations with $\mathfrak{f}$-projectors.

After this introduction of $\mathfrak{f}$-normalizers we are able to give a construction of a maximal S'_w-closed local definition under a certain hypothesis on $\mathfrak{f}$. The closure operation S'_w is just the analogue of the well-placed subgroups closure operation S_w, but using the generalized Fitting subgroup $F'(G) = \mathrm{Soc}(G \bmod \Phi(G))$. The construction of the maximal local definition of a

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saturated formation of soluble groups was done by K. Doerk in [4]. In [12], this problem is investigated by P. Förster and E. Salomon in the general case. We give some sufficient conditions for a saturated formation of finite groups to have a maximal local definition.

2. Preliminaries

In this section, we collect some definitions and notations as well as some well-known elementary results, omitting their proofs.

First recall that a *primitive* group is a group G such that for some maximal subgroup U of G, $U_G = 1$ (where U_G is the intersection of all G-conjugates of U, the largest unique normal subgroup of G contained in U).

A primitive group is of one of the following types:

(1) $\mathrm{Soc}(G)$, the socle of G, is an abelian minimal normal subgroup of G, complemented by U.

(2) $\mathrm{Soc}(G)$ is a non-abelian minimal normal subgroup of G.

(3) $\mathrm{Soc}(G)$ is the direct product of the two minimal normal subgroups of G which are both non-abelian and complemented by U.

We will denote by $\mathfrak{P}$ the class of all primitive groups and by $\mathfrak{P}_i$, $i \in \{1, 2, 3\}$ the class of all primitive groups of type i.

For basic properties of the primitive groups, the reader is referred to [10].

If H/K is a chief factor of G such that $H/K \leqq \Phi(G/K)$, then H/K is said to be a Frattini chief factor of G. If H/K is not a Frattini chief factor of G, then it is supplemented by a maximal subgroup U of G (i.e. $G = UH$ and $K \leqq U \cap H$).

A subgroup H of a group G is called CAP-subgroup (Cover and Avoidance Property), if for every chief factor R/K of G, H either covers ($R = K(R \cap H)$) or avoids ($R \cap H \leqq K$) it.

For more details about formations and Schunck classes the reader is referred to [3], [6], [10], [11]. The notation is standard and can be found mainly in [14].

All groups considered here are supposed to belong to a fixed but otherwise arbitrary universe $\mathfrak{V}$ contained in $\mathfrak{E}$, the class of all finite groups, such that $\mathfrak{V} = \{Q, S, R_0, E_\Phi\}\mathfrak{V}$.

All classes of groups considered will be $\mathfrak{V}$-classes, i.e. if $\mathfrak{X}$ is a class of groups we suppose that $\mathfrak{X}$ is contained in $\mathfrak{V}$.

3. $\mathfrak{h}$-Critical subgroups

(3.1) Definition. Let M be a maximal subgroup of a group G. Then the group $X = G/M_G$ is a primitive group; we say that M is of *type i* if $X \in \mathfrak{P}_i$

($1 \leqq i \leqq 3$) and M is a *monolithic maximal subgroup* of G if M is of type 1 or type 2.

(3.2.) DEFINTION. Given a Schunck class $\mathfrak{h}$, a maximal subgroup U of a group G is called $\mathfrak{h}$-*normal* in G if $G/U_G \in \mathfrak{h}$ and $\mathfrak{h}$-*abnormal* otherwise.

(3.3) DEFINITION. Let U, G and $\mathfrak{h}$ be as above. U is $\mathfrak{h}$-*critical* in G, if U is $\mathfrak{h}$-abnormal monolithic maximal subgroup of G and $G = UF'(G)$ where $F'(G) = \mathrm{Soc}(G \bmod \Phi(G))$. For properties of $F'(G)$ see [11].

It is not difficult to prove:

(3.4) LEMMA. *If U is $\mathfrak{h}$-critical in G and N is a normal subgroup of G such that $N \leqq U$, then U/N is $\mathfrak{h}$-critical in G/N.*

We want to describe all Schunck classes with the following property:

(C) If $G \notin \mathfrak{h}$, then G contains an $\mathfrak{h}$-critical subgroup.

Property (C) is not satisfied by all Schunck classes. For instance, let $\mathfrak{h}$ be the Schunck class generated by a non-abelian simple group S and $\mathfrak{B} = \mathfrak{E}$. Then $G = S \times S \in b(\mathfrak{h})$ and G does not contain $\mathfrak{h}$-critical subgroups.

Förster in [9; 2.14] characterizes all Schunck classes with the property (C) in the soluble case. The same characterization holds in the general case, although here we must deal with non-soluble primitive groups; the proof is similar to Förster's.

(3.5) THEOREM. *For a Schunck class $\mathfrak{h}$, the following three statements are pairwise equivalent*:

(i) $\mathfrak{h}$ *has the property* (C).

(ii) $\mathfrak{h} = E_\Phi Q R_0 \mathrm{Pr}(\mathfrak{h})$ *with* $\mathrm{Pr}(\mathfrak{h}) = \mathfrak{h} \cap \mathfrak{B}$.

(iii) $\mathfrak{h} = E_\Phi \mathfrak{f}$ *for some formation* $\mathfrak{f}$.

(3.6) DEFINITIONS. (a) [10] Let H/K be a chief factor of G. Denote:

$$[H/K]^*G = \begin{cases} [H/K](G/C_G(H/K) & \text{if } H/K \text{ is abelian,} \\ G/C_G(H/K) & \text{if } H/K \text{ is non-abelian.} \end{cases}$$

The primitive group $[H/K]^*G$ is the *monolithic primitive group associated with the chief factor H/K of G.*

Note that if H/K is a non-Frattini chief factor of G and M is a monolithic maximal subgroup of G supplementing H/K in G, then $G/M_G \cong [H/K]^*G$.

(b) Given a Schunck class $\mathfrak{h}$, a chief factor H/K of a group G is said to be $\mathfrak{h}$-*central* in G if $[H/K]^*G \in \mathfrak{h}$ and $\mathfrak{h}$-*eccentric* otherwise.

4. $\mathfrak{h}$-Normalizers

We assume that $\mathfrak{h}$ is a Schunck $\mathfrak{B}$-class of the form $\mathfrak{h} = E_\Phi \mathfrak{f}$ for some formation $\mathfrak{f}$. Thus, the existence of $\mathfrak{h}$-critical subgroups is assured in every group $G \in \mathfrak{B} - \mathfrak{h}$.

This allows us to define $\mathfrak{h}$-normalizers in every group G of the universe $\mathfrak{B}$ in an abstract way.

(4.1) Definition. Let G be a group in $\mathfrak{B}$. A subgroup D of G is an $\mathfrak{h}$-normalizer of G, if there exists a chain of subgroups:

$$D = H_n \leqq H_{n-1} \leqq \cdots \leqq H_1 \leqq H_0 = G \tag{1}$$

such that H_i is an $\mathfrak{h}$-critical subgroup of H_{i-1} $(i = 1, \ldots, n)$ and such that H_n contains no $\mathfrak{h}$-critical subgroup.

If $G \in \mathfrak{h}$, we interpret the definition to mean $D = G$. The condition on H_n is equivalent to $D \in \mathfrak{h}$.

Denote by $\mathrm{Nor}_\mathfrak{h}(G)$ the set of all $\mathfrak{h}$-normalizers of G.

If $\mathfrak{B} = \mathfrak{S}$, the class of soluble groups, this definition coincides with the classical one in [3], [15].

Of course $\mathfrak{h}$-normalizers are invariant under epimorphisms.

(4.2) Proposition. *Let D be an $\mathfrak{h}$-normalizer of a group G and N a normal subgroup of G; then DN/N is an $\mathfrak{h}$-normalizer of G/N.*

It is not true in general that $\mathrm{Nor}_\mathfrak{h}(G)$ is a conjugacy class of subgroups of G. For instance, take $\mathfrak{B} = \mathfrak{E}$ and $\mathfrak{h} = \mathfrak{N}$ the class of nilpotent groups. The $\mathfrak{N}$-critical subgroups of Alt(5) are isomorphic to Alt(4), Dih(10) and Sym(3). Thus, we obtain two distinct conjugacy classes of $\mathfrak{N}$-normalizers, isomorphic to C_3 and C_2.

This example also shows that we cannot talk in general of the cover-avoidance property.

If M is an $\mathfrak{h}$-critical subgroup of G and H/K an $\mathfrak{h}$-central chief factor of G, then M covers it and $[H \cap M/K \cap M]^*M \cong [H/K]^*G$. If H/K is a non-Frattini chief factor of G covered by M, then it is easy to see that $H \cap M/K \cap M$ is a chief factor of M and $\mathrm{Aut}_G(H/K) \cong \mathrm{Aut}_M(H \cap M/K \cap M)$.

Repeated use of these facts, together with $D \in \mathfrak{h}$, proves easily the following:

(4.3) THEOREM. *Let G be a group and $D \in \mathrm{Nor}_{\mathfrak{h}}(G)$.*

(i) *If H/K is an $\mathfrak{h}$-central chief factor of G, then D covers H/K and $H \cap D/K \cap D$ is a chief factor of D and $\mathrm{Aut}_G(H/K)$ is isomorphic to $\mathrm{Aut}_D(H \cap D/K \cap D)$.*

(ii) *Among the non-Frattini chief factors of G, D covers exactly the $\mathfrak{h}$-central ones.*

Unfortunately, nothing can be said on the $\mathfrak{h}$-eccentric chief factors of G.

EXAMPLE 1. Take $\mathfrak{B} = \mathfrak{E}$ and the Schunck class $\mathfrak{h} = E_{\Phi}\mathfrak{N}^*$ where $\mathfrak{N}^*$ denotes the class of quasinilpotent groups, i.e. $\mathfrak{h} = (G/F'(G) = G)$.

Alt(5) has an irreducible and faithful module M over GF(2). Let $X = [M]A$ with A isomorphic to Alt(5). We have that $X \in \mathfrak{B}_1$ and $M = F'(X) = \mathrm{Soc}(X)$. Thus $X \notin \mathfrak{h}$. Now, A is $\mathfrak{h}$-critical in X and $A \in \mathrm{Nor}_{\mathfrak{h}}(X)$. All chief factors of X are non-Frattini and A covers X/M. Now, X/M is an $\mathfrak{h}$-central chief factor of X. Moreover A avoids M which is an $\mathfrak{h}$-eccentric chief factor of X.

Next, we consider the relation of the $\mathfrak{h}$-normalizers to the maximal subgroups of G.

(4.4) THEOREM. *Let M be a monolithic maximal subgroup of a group G. Then M contains an $\mathfrak{h}$-normalizer of G if and only if M is $\mathfrak{h}$-abnormal in G.*

PROOF. It is easy to see that if M is a maximal subgroup of G containing an $\mathfrak{h}$-normalizer of G, then M is $\mathfrak{h}$-abnormal in G. Conversely, let M be an $\mathfrak{h}$-abnormal monolithic maximal subgroup of G. Denote $R = \mathrm{Soc}(G \bmod M_G)$. If M is $\mathfrak{h}$-critical in G, the result is obvious. Otherwise, $F'(G) \leqq M_G$. Let X be an $\mathfrak{h}$-critical subgroup of G. It is clear that M_G is not contained in X. Then, $G = XM_G$ and $R = M_G(R \cap X)$. Since $M_G \leqq C_G(R/M_G)$ we have $G = XC_G(R/M_G)$. Thus, $R \cap X/M_G \cap X$ is a chief factor of X and $\mathrm{Aut}_X(R \cap X/M_G \cap X)$ is isomorphic to $\mathrm{Aut}_G(R/M_G)$. Consequently, $R \cap X/M_G \cap X$ is an $\mathfrak{h}$-eccentric chief factor of X.

On the other hand, $M \cap X$ is a maximal subgroup of X. Since $M_G \cap X = (M \cap X)_X$ and $X = (M \cap X)(R \cap X)$ we have $X/(M \cap X)_X \notin \mathfrak{h}$. Then, $M \cap X$ is an $\mathfrak{h}$-abnormal monolithic maximal subgroup of X. By induction, $M \cap X$ contains an $\mathfrak{h}$-normalizer of X. Since $\mathrm{Nor}_{\mathfrak{h}}(X)$ is contained in $\mathrm{Nor}_{\mathfrak{h}}(G)$, we obtain the stated result.

It is not true in general that an $\mathfrak{h}$-abnormal maximal subgroup M of a group G contains an $\mathfrak{h}$-normalizer of G. For instance, take the saturated formation $\mathfrak{h} = (T : \mathrm{Alt}(5) \notin Q(T))$ and $\mathfrak{B} = \mathfrak{E}$. Consider $G = A \times B$ the direct product of A

and B, where $A \cong B \cong \mathrm{Alt}(5)$. If U is a maximal subgroup of G such that $U_G = 1$, then U is an $\mathfrak{h}$-abnormal maximal subgroup of G that does not contain any $\mathfrak{h}$-normalizer of G. Suppose, arguing by contradiction, that there exists $E \in \mathrm{Nor}_{\mathfrak{h}}(G)$ such that $E \leqq U$. Let M be an $\mathfrak{h}$-critical subgroup of G with $E \leqq M$ and $E \in \mathrm{Nor}_{\mathfrak{h}}(M)$. Since M is monolithic, we can assume $M_G = A$. Therefore, $M = (M \cap B) \times A$. Let S be a minimal normal subgroup of M contained in $M \cap B$. Clearly, S is a non-Frattini $\mathfrak{h}$-central chief factor of M. By (4.3), E covers S. Consequently, $S \leqq M \cap B \cap U = 1$, a contradiction.

In the rest of this Section 4 assume that $\mathfrak{f}$ is a saturated $\mathfrak{B}$-formation. Most of the properties of $\mathfrak{f}$-normalizers of soluble groups, such as conjugacy, cover-avoidance property, relation with $\mathfrak{f}$-projectors, do not hold in the general case. However, $\mathfrak{f}$-normalizers of groups G such that $G^{\mathfrak{f}}$ is soluble (i.e. $G \in \mathfrak{S}\mathfrak{f}$) do really verify those classical properties.

(4.5) Theorem. *Let G be a group such that $G^{\mathfrak{f}} \in \mathfrak{S}$. Then*:

(i) *If D is an $\mathfrak{f}$-normalizer of G, D is a* CAP-*subgroup of G that covers the $\mathfrak{f}$-central chief factors of G and avoids the $\mathfrak{f}$-eccentric ones.*

(ii) *Let H be a subgroup of G with $G = HF(G)$. Then, there exists $A \in \mathrm{Proj}_{\mathfrak{f}}(H)$ and $E \in \mathrm{Proj}_{\mathfrak{f}}(G)$ such that $A = H \cap E$.*

(iii) *Every $\mathfrak{f}$-normalizer of G is contained in an $\mathfrak{f}$-projector of G.*

(iv) *For every Hall system Σ of $G^{\mathfrak{f}}$, every $\mathfrak{f}$-projector of $N_G(\Sigma)$ is an $\mathfrak{f}$-normalizer of G. Thus, $\bigcup \{\mathrm{Proj}_{\mathfrak{f}}(N_G(\Sigma))/\Sigma$ Hall system of $G^{\mathfrak{f}}\} = \mathrm{Nor}_{\mathfrak{f}}(G)$.*

(v) $\mathrm{Nor}_{\mathfrak{f}}(G)$ *is a conjugacy class of subgroups of G.*

Proof. (i) We use induction on the order of G. Let D be an $\mathfrak{f}$-normalizer of G and assume that D is a maximal subgroup of G. If H/K is a chief factor of G and H/K is non-abelian, D covers H/K because D is a maximal of type 1. If H/K is abelian and D does not cover H/K, HD_G/D_G is a minimal normal subgroup of G/D_G and $D_G(H \cap D) = D_G$. Then, $H \cap D = K$ and D avoids H/K.

If D is not a maximal subgroup of G, then there exists an $\mathfrak{f}$-critical maximal subgroup M of G such that $D \leqq M$, $D \in \mathrm{Nor}_{\mathfrak{f}}(M)$ and $G = MF(G)$. By induction, D is a CAP-subgroup of M. Now, M is a CAP-subgroup of G and then D is a CAP-subgroup of G (see [8] lemma (4.4)).

If H/K is an $\mathfrak{f}$-central chief factor of G, by (4.3) D covers H/K. Suppose H/K is an $\mathfrak{f}$-eccentric chief factor of G. If D covers H/K, then $H \cap D/K \cap D$ is a chief factor of D and we have $[H/K]*G \cong [H \cap D/K \cap D]*D$. Now, D is an $\mathfrak{f}$-

group and then all chief factors of D are $\mathfrak{f}$-central. Thus, $[H/K]^*G \in \mathfrak{f}$, a contradiction. Therefore, H/K must be avoided by D.

Using similar arguments to those used in [3; 5.12] (ii) is proved.

(iii) We use induction on $|G|$. We can suppose $G \notin \mathfrak{f}$. Let D be an $\mathfrak{f}$-normalizer of G. Then, there exists an $\mathfrak{f}$-critical subgroup M of G such that $D \leqq M$ and $D \in \mathrm{Nor}_{\mathfrak{f}}(M)$. Since $M^{\mathfrak{f}}$ is soluble, there exists $A \in \mathrm{Proj}_{\mathfrak{f}}(M)$ such that $D \leqq A$. Now, $G = MF(G)$. By (ii), there exists $B \in \mathrm{Proj}_{\mathfrak{f}}(M)$ and $E \in \mathrm{Proj}_{\mathfrak{f}}(G)$ such that $B = M \cap E$. Since $M^{\mathfrak{f}}$ is soluble, by corollary (5.3) of [16] A and B are conjugate in M, i.e., $A = B^m$ with $m \in M$. Then $A = M \cap E^m$ and D is contained in $E^m \in \mathrm{Proj}_{\mathfrak{f}}(G)$.

Using the above properties and with similar arguments to those used in [18], (iv) is proved.

(v) Let Σ be a Hall system of $G^{\mathfrak{f}}$. Then, $N_G(\Sigma)$ is an $\mathfrak{N}\mathfrak{f}$-group. By [16; 5.3] two $\mathfrak{f}$-projectors of $N_G(\Sigma)$ are conjugated. On the other hand, two Hall systems of $G^{\mathfrak{f}}$ are conjugated. Applying (iv), $\mathrm{Nor}_{\mathfrak{f}}(G)$ is a conjugacy class of subgroups of G.

Example 2. We take $\mathfrak{B} = \mathfrak{E}$ and the Schunck class $\mathfrak{h} = E_\Phi \mathfrak{N}^*$ where $\mathfrak{N}^*$ denotes, as in Example 1, the class of quasinilpotent groups. If V is a 3-dimensional vector space over GF(2) and $G = [V]\mathrm{Aut}\, V$, then $G/G_{\mathfrak{S}} \in \mathfrak{h}$ and $\mathrm{Nor}_{\mathfrak{h}}(G)$ is the set of $\mathfrak{h}$-maximal supplements of Soc(G). But an example on p. 161 of [14] shows that they are not all conjugate. This example shows that conjugacy of $\mathfrak{h}$-normalizers does not hold in groups G such that $G/G_{\mathfrak{S}} \in \mathfrak{h}$, where $\mathfrak{h}$ is a Schunck class not a saturated formation.

Let G be a group with $G^{\mathfrak{f}}$ soluble. Since $\mathrm{Proj}_{\mathfrak{f}}(G)$ is a conjugacy class of subgroups of G, every $\mathfrak{f}$-projector of G contains an $\mathfrak{f}$-normalizer of G. This property is not true in general. If we take $\mathfrak{B} = \mathfrak{E}$ and $\mathfrak{f} = \mathfrak{N}$, the class of nilpotent groups, the group $G = \mathrm{Alt}(5)$ has three distinct conjugacy classes of $\mathfrak{N}$-projectors, namely, $\mathrm{Syl}_p(G)$, $p = 2, 3, 5$. Hence, if $P \in \mathrm{Syl}_5(G)$ then P does not contain any $\mathfrak{N}$-normalizer of G.

Now we can follow [3; 5.15] and with little changes we are able to prove the following complementation theorem which is a generalization of one due to G. Higman.

(4.6) Theorem. *Let G be a group such that $G^{\mathfrak{f}}$ is abelian. Then, $G^{\mathfrak{f}}$ is complemented in G and any two complements in G are conjugate. The complements are the $\mathfrak{f}$-normalizers of G.*

Next, we use (4.6) to give a short proof of a well-known result of Semetkov (cf. [17]).

(4.7) THEOREM (Semetkov). *Let G be a group such that for some prime p, the Sylow p-subgroups of $G^{\mathfrak{f}}$ are abelian. Then every chief factor of G below $G^{\mathfrak{f}}$ whose order is divisible by p is an $\mathfrak{f}$-eccentric chief factor of G.*

PROOF. Suppose the theorem is false and let G be a minimal counter-example. Then $G^{\mathfrak{f}} \neq 1$. Let N be a minimal normal subgroup of G such that $N \leqq G^{\mathfrak{f}}$. From minimality of G, every chief factor of G between N and $G^{\mathfrak{f}}$ whose order is divisible by the prime p is $\mathfrak{f}$-eccentric and N is the unique minimal normal subgroup of G contained in $G^{\mathfrak{f}}$. Then p divides $|N|$ and N is an $\mathfrak{f}$-central chief factor of G. Since $G/C_G(N) \in \mathfrak{f}$, $N \leqq Z(G^{\mathfrak{f}})$ and N is an abelian p-group. Let P be a Sylow p-subgroup of G such that $N \leqq P$. If $(G^{\mathfrak{f}})' \neq 1$, then $N \leqq (G^{\mathfrak{f}})' \cap Z(G^{\mathfrak{f}}) \cap P = 1$ by [14; 2.2 Satz p. 416], a contradiction. Thus $(G^{\mathfrak{f}})' = 1$ and $G^{\mathfrak{f}}$ is an abelian group. Applying (4.6) $G^{\mathfrak{f}}$ is complemented in G by an $\mathfrak{f}$-normalizer. By (4.5), N is an $\mathfrak{f}$-eccentric chief factor of G, a contradiction.

Schmid, in [16], proves the following theorem:

(4.8) THEOREM (Schmid). *Let G be a group such that every chief factor of G below $G^{\mathfrak{f}}$ is $\mathfrak{f}$-eccentric. If $G^{\mathfrak{f}}$ is p-nilpotent for every prime p in $\pi = \pi\,|G : G^{\mathfrak{f}}|$, then $G^{\mathfrak{f}}$ is complemented in G and any two complements are conjugate.*

Next, we use our normalizers to analyze the complements of $G^{\mathfrak{f}}$ in this theorem.

(4.9) THEOREM. *Let G be a group as above. The complements of $G^{\mathfrak{f}}$ are precisely the $(\mathfrak{f} \cap \mathfrak{E}_\pi)$-normalizers of G (here, $\mathfrak{E}_\pi$ denotes the class of π-groups).*

PROOF. We prove by induction on the order of G that every $(\mathfrak{f} \cap \mathfrak{E}_\pi)$-normalizer is a complement of $G^{\mathfrak{f}}$ in G. First, we note that $\mathfrak{L} = \mathfrak{f} \cap \mathfrak{E}_\pi$ is a saturated formation and $G^{\mathfrak{L}} = G^{\mathfrak{f}}$. Let N be the normal π-complement of $G^{\mathfrak{f}}$. Then, $G^{\mathfrak{f}}/N$ is a nilpotent π-group. If $N \neq 1$ and E is an $\mathfrak{L}$-normalizer of G, then EN/N is an $\mathfrak{L}$-normalizer of G/N. By induction, $E \cap G^{\mathfrak{f}} \leqq N$. Since E is a π-group and N is a π'-group, we have that $E \cap G^{\mathfrak{f}} = 1$ and the theorem is proved. Thus, we can suppose that $N = 1$ and then $G^{\mathfrak{f}}$ is a nilpotent π-group. Consequently, G is a π-group and every $\mathfrak{L}$-normalizer of G is an $\mathfrak{f}$-normalizer of G. Since $G^{\mathfrak{f}}$ is a nilpotent group, every $\mathfrak{f}$-normalizer E of G avoids every $\mathfrak{f}$-eccentric chief factor of G. Then, $E \cap G^{\mathfrak{f}} = 1$.

Now, if T is a complement of $G^{\mathfrak{f}}$ in G, T is conjugate to an $\mathfrak{L}$-normalizer of G. Consequently, T is an $\mathfrak{L}$-normalizer of G.

5. Some facts about formations

(5.1) Definitions. (a) We call a subgroup E of a group $G \in \mathfrak{V}$ *well-placed* in G, if there exists a chain:

$$E = E_n \leqq E_{n-1} \leqq \cdots \leqq E_0 = G, \quad \text{such that } E_{i-1} = E_i F'(E_{i-1}) \text{ for every } i.$$

We let S'_w be the closure operation defined by:

$S'_w \mathfrak{X} = (E : E$ is a well-placed subgroup of an $\mathfrak{X}$-group), for every group class $\mathfrak{X}$.

(b) A formation function $g = \{g(p) : p$ a prime number$\}$ is said to be S'_w-closed if $g(p)$ is an S'_w-closed formation, for every prime p.

In the soluble case, $S'_w = S_w$ and every formation is S_w-closed ([2; 1.8]).

Let $\mathfrak{h}$ be a $\mathfrak{V}$-Schunck class of the form $\mathfrak{h} = E_\Phi \mathfrak{f}$ for some formation $\mathfrak{f}$. The $\mathfrak{h}$-critical maximal subgroups and the $\mathfrak{h}$-normalizers of a group G are both examples of well-placed subgroups. Moreover, it is not difficult to prove that if N is a normal Hall π-subgroup of a group $G \in \mathfrak{V}$ and X is a complement to N in G, then X is a well-placed subgroup of G (see [13]).

Formations are not S'_w-closed in general. For instance, let $\mathfrak{N}^*$ be the formation of quasinilpotent groups and $\mathfrak{V} = \mathfrak{E}$. Every subgroup of Alt(5) is well-placed in Alt(5). If H is a subgroup of Alt(5) isomorphic to Dih(10), then $H \in S'_w \mathfrak{N}^* - \mathfrak{N}^*$. Hence, $\mathfrak{N}^*$ is not S'_w-closed.

(5.2) Definition. Let $\mathfrak{h}$ be an Schunck $\mathfrak{V}$-class of the form $E_\Phi \mathfrak{f}$ for some formation $\mathfrak{f}$ and let $\mathfrak{R}$ be a $\mathfrak{V}$-formation.

Denote $\mathfrak{A} = (G \in \mathfrak{V}/\mathrm{Nor}_{\mathfrak{h}}(G) \subset \mathfrak{R})$. We prove that $\mathfrak{A}$ is an R_0-closed class. Let $i \in \{1, 2\}$ and let $G/N_i \in \mathfrak{A}$ with $N_1 \cap N_2 = 1$. If $D \in \mathrm{Nor}_{\mathfrak{h}}(G)$, then $DN_i/N_i \in \mathrm{Nor}_{\mathfrak{h}}(G/N_i)$. Hence, $D/D \cap N_i \in \mathfrak{R}$ and $D \in R_0 \mathfrak{R} = \mathfrak{R}$. Consequently, $R_0 \mathfrak{A} = \mathfrak{A}$. Now, denote $\mathfrak{h}_{\mathfrak{R}} = Q\mathfrak{A}$. Then, we have $R_0 \mathfrak{h}_{\mathfrak{R}} = R_0 Q \mathfrak{A} \subset Q R_0 \mathfrak{A} = Q\mathfrak{A} = \mathfrak{h}_{\mathfrak{R}}$. Thus, $\mathfrak{h}_{\mathfrak{R}}$ is R_0-closed. Consequently, $\mathfrak{h}_{\mathfrak{R}}$ is a $\mathfrak{V}$-formation containing $\mathfrak{h} \cap \mathfrak{R}$.

(5.3) Definition. Let $\mathfrak{f}$ be a saturated $\mathfrak{V}$-formation locally defined by an integrated and full formation function f.

For every prime p, denote by $f^*(p)$ the formation

$$f^*(p) = Q(G : \mathrm{Nor}_{\mathfrak{f}}(G) \subset f(p))$$

(i.e. $f^*(p) = \mathfrak{f}_{f(p)}$ in the notation of (5.2)).

A group $G \in b(\mathfrak{f})$ is called *strongly dense* (with respect to $\mathfrak{f}$) if $G \in f^*(p)$ for each prime $p \in \pi(\mathrm{Soc}(G))$.

The boundary $b(\mathfrak{f})$ is said to be *strongly wide* if it does not contain any strongly dense group.

Recall that a group $G \in b(\mathfrak{f})$ is dense with respect to $\mathfrak{f}$ if $G \in b(f(p))$ for each prime $p \in \pi(\mathrm{Soc}(G))$. The boundary $b(\mathfrak{f})$ is said to be wide if it does not contain any dense group (see [12]).

(5.4) Remark. If a group G is strongly dense with respect to $\mathfrak{f}$, then G is dense with respect to $\mathfrak{f}$. Let G be a group in $b(\mathfrak{f})$ such that G is strongly dense with respect to $\mathfrak{f}$. Then, for each prime $p \in \pi(\mathrm{Soc}(G))$, there exists $T(p) \in \mathrm{Nor}_{\mathfrak{f}}(G)$ such that $T(p) \in f(p)$. Since $G/\mathrm{Soc}(G) \in \mathfrak{f}$, $\mathrm{Nor}_{\mathfrak{f}}(G/\mathrm{Soc}(G)) = \{G/\mathrm{Soc}(G)\}$ and $G = T(p)\mathrm{Soc}(G)$. Then, $G/\mathrm{Soc}(G) \in f(p)$. Consequently, $G \in b(f(p))$, for each $p \in \pi(\mathrm{Soc}(G))$ and G is dense with respect to $\mathfrak{f}$.

The converse is not true in general. Take $\mathfrak{B} = \mathfrak{E}$, the class of all finite groups, and let $\mathfrak{N}$ be the class of nilpotent groups. The integrated and full formation function f such that $\mathfrak{N} = \mathrm{LF}(f)$ is done by $f(p) = \mathfrak{S}_p$ for every prime p, where $\mathfrak{S}_p$ denotes the class of p-groups. Then, $G = \mathrm{Alt}(5)$ is dense with respect to $\mathfrak{N}$, but G is not strongly dense with respect to $\mathfrak{N}$. In fact, $G \notin f^*(5)$.

Let G be a group and H/K be a chief factor of G. Denote by $C_G^*(H/K)$ the set of all elements $g \in G$ such that conjugation by gK induces an inner automorphism in H/K.

Recall the definitions of the class of nilpotent groups:

$$\mathfrak{N} = \{G \in \mathfrak{E} \mid \text{every chief factor } H/K \text{ of } G \text{ verifies } G = C_G(H/K))$$

and the class of quasinilpotent groups:

$$\mathfrak{N}^* = (G \in \mathfrak{E} \mid \text{every chief factor } H/K \text{ of } G \text{ verifies } G = G_G^*(H/K)).$$

In a similar way, if $\mathfrak{f}$ is a saturated $\mathfrak{B}$-formation locally defined by a formation function f, we can define:

$$\mathfrak{f}^* = \{G \in \mathfrak{B} \mid \text{every chief factor } H/K \text{ of } G \text{ verifies } G/C_G^*(H/K) \in f(p) \\ \text{for each prime } p \text{ dividing the order of } H/K).$$

(5.5) Proposition. *The following statements are equivalent*:

(i) $\mathfrak{f} = \mathfrak{f}^*$.

(ii) $b(\mathfrak{f})$ *is wide*.

Proof. (i) implies (ii). Suppose there exists a group $G \in b(\mathfrak{f})$ such that G is dense with respect to $\mathfrak{f}$. Then, G is a monolithic primitive group and for each $p \in \pi(\mathrm{Soc}(G))$ we have $G \in b(f(p))$. Since $\mathrm{Soc}(G) = C_G^*(\mathrm{Soc}(G))$,

$G/C_G^*(\mathrm{Soc}(G)) \in f(p)$ for each $p \in \pi(\mathrm{Soc}(G))$. But this implies that $G \in \mathfrak{f}^* = \mathfrak{f}$, a contradiction. Thus, $b(\mathfrak{f})$ is wide.

(ii) implies (i). It is clear that $\mathfrak{f} \subset \mathfrak{f}^*$. Suppose $\mathfrak{f}^* \neq \mathfrak{f}$ and let G be a group in $\mathfrak{f}^* - \mathfrak{f}$ of least order. Then $G \in b(\mathfrak{f})$. Since $G \in \mathfrak{f}^*$ we have $G/C_G^*(\mathrm{Soc}(G)) \in f(p)$ for each $p \in \pi(\mathrm{Soc}(G))$. Hence $G/\mathrm{Soc}(G) \in f(p)$ for each $p \in \pi(\mathrm{Soc}(G))$. This is to say, $G \in b(f(p))$ for each $p \in \pi(\mathrm{Soc}(G))$ and $b(\mathfrak{f})$ is not wide, a contradiction.

(5.6) Corollary. *Let $\mathfrak{f}$ be a saturated $\mathfrak{B}$-formation. If $\mathfrak{f}$ contains all nilpotent groups in $\mathfrak{B}$, and $b(\mathfrak{f})$ is wide, then $\mathfrak{f}$ contains all quasinilpotent groups in $\mathfrak{B}$.*

6. Local formations

For any group class $\mathfrak{X}$, and for any closure operation C, $\mathfrak{X}^C$ denotes the largest C-closed class contained in $\mathfrak{X}$, whenever such a class exists.

Let $\mathfrak{f}$ be a saturated $\mathfrak{B}$-formation and $\mathfrak{h}$ a $\mathfrak{B}$-formation. Let $\mathfrak{f}_{\mathfrak{h}}$ be the $\mathfrak{B}$-formation defined as (5.2).

(6.1) Lemma. *Let $\mathfrak{X}$ be an S'_w-closed formation. Then, $\mathfrak{X}$ is contained in $\mathfrak{f}_{\mathfrak{h}}$ if and only if $\mathfrak{f} \cap \mathfrak{X} \subset \mathfrak{h}$. Thus, if $\mathfrak{C} = (\mathfrak{f}_{\mathfrak{h}})^{\{QR_0, S'_w\}}$, $\mathfrak{C}$ is the largest S'_w-closed formation such that $\mathfrak{f} \cap \mathfrak{C} \subset \mathfrak{h}$.*

Proof. Suppose that $\mathfrak{X}$ is an S'_w-closed formation such that $\mathfrak{X} \subset \mathfrak{f}_{\mathfrak{h}}$. Let G be a group in $\mathfrak{f} \cap \mathfrak{X}$. Then, there exists a group R such that $\mathrm{Nor}_{\mathfrak{f}}(R) \subset \mathfrak{h}$ and there exists a normal subgroup N of R with $G \cong R/N$. If $D \in \mathrm{Nor}_{\mathfrak{f}}(R)$, $DN/N \in \mathrm{Nor}_{\mathfrak{f}}(R/N)$. Since $R/N \in \mathfrak{f}$, we have $DN/N = R/N$. Hence, $G \in \mathfrak{h}$ and we have $\mathfrak{f} \cap \mathfrak{X} \subset \mathfrak{h}$.

Conversely, take $G \in \mathfrak{X}$ and $D \in \mathrm{Nor}_{\mathfrak{f}}(G)$. Since $\mathfrak{X}$ is S'_w-closed, $D \in S'_w\mathfrak{X} = \mathfrak{X}$. Then, $D \in \mathfrak{f} \cap \mathfrak{X} \subset \mathfrak{h}$ and $G \in \mathfrak{f}_{\mathfrak{h}}$. Thus, $\mathfrak{X}$ is contained in $\mathfrak{f}_{\mathfrak{h}}$. .

In the following, $\mathfrak{f}$ will be a saturated $\mathfrak{B}$-formation and f the integrated and full formation function such that $\mathfrak{f} = \mathrm{LF}(f)$.

(6.2) Theorem. *Let g be an S'_w-closed formation function. Then, $\mathfrak{f} = \mathrm{LF}(g)$ if and only if the following two conditions hold:*

(a) If $G \in b(\mathfrak{f})$ is strongly dense with respect to $\mathfrak{f}$, then $G \notin g(p)$ for some $p \in \pi(\mathrm{Soc}(G))$.

(b) $f_0 \leqq g \leqq f^*$, where f_0 is the minimal formation function such that $\mathfrak{f} = \mathrm{LF}(f_0)$.

PROOF. First, we note that $f_0 \leqq f^*$. Suppose that $\mathfrak{f} = \mathrm{LF}(g)$ and for every prime p, $S'_w g(p) = g(p)$. Then, every strongly dense group $G \in b(\mathfrak{f})$ such that $G \in g(p)$ for each $p \in \pi(\mathrm{Soc}(G))$ belongs to $\mathfrak{f}$, and so (a) holds.

Since f_0 is the minimal formation function such that $\mathfrak{f} = \mathrm{LF}(f_0)$, we have $f_0 \leqq g$. Moreover, if $h(p) = \mathfrak{S}_p(g(p) \cap \mathfrak{f})$ for every prime p, h is an integrated and full formation function such that $\mathfrak{f} = \mathrm{LF}(h)$. Since f is unique, $f(p) = h(p)$ for each prime p. Then, for every prime p, we have $g(p) \cap \mathfrak{f} \subset f(p)$. Applying the above lemma, $g(p) \subset f^*(p)$. Thus $f_0 \leqq g \leqq f^*$.

Conversely, suppose g satisfies (a) and (b). Then, it remains to show that $\mathrm{LF}(g) \subset \mathfrak{f}$ since then $\mathfrak{f} = \mathrm{LF}(f_0) \subset \mathrm{LF}(g) \subset \mathfrak{f}$. Consider a group $G \in \mathrm{LF}(g) - \mathfrak{f}$ of least order. Then, $G \in b(\mathfrak{f})$ and G is a monolithic primitive group. If $\mathrm{Soc}(G)$ is abelian of characteristic p, say, then we deduce that $G/C_G(\mathrm{Soc}(G)) \in g(p) \cap \mathfrak{f}$. Since $g(p)$ is an S'_w-closed formation and $g(p) \subset f^*(p)$ we have $g(p) \cap \mathfrak{f} \subset f(p)$. Now, $G/\mathrm{Soc}(G) \in \mathfrak{f}$. Then, $G \in \mathfrak{f}$, a contradiction. Hence, $\mathrm{Soc}(G)$ is non-abelian and we have $G \in g(p) \subset f^*(p)$ for each $p \in \pi(\mathrm{Soc}(G))$. This implies that G is strongly dense with respect to $\mathfrak{f}$ and $G \in g(p)$ for each $p \in \pi(\mathrm{Soc}(G))$, which contradicts (a). Thus, $\mathfrak{f} = \mathrm{LF}(g)$.

(6.3) PROPOSITION. *The following statements are equivalent*:
(a) $b(\mathfrak{f})$ *is strongly wide*.
(b) $\mathfrak{f} = \mathrm{LF}(f^*)$.

PROOF. Since $f_0 \leqq f^*$, we have $\mathfrak{f} = \mathrm{LF}(f_0) \subset \mathrm{LF}(f^*)$.

(a) implies (b). Suppose that $\mathfrak{f} \neq \mathrm{LF}(f^*)$ and choose a group G in $\mathrm{LF}(f^*) - \mathfrak{f}$ of least order. Then, $G \in b(\mathfrak{f})$ and for every $p \in \pi(\mathrm{Soc}(G))$ we have $G/C_G(\mathrm{Soc}(G)) \in f^*(p)$. If $1 \neq C_G(\mathrm{Soc}(G))$, then $\mathrm{Soc}(G)$ is abelian of characteristic p, say. Since $G/C_G(\mathrm{Soc}(G)) \in f^*(p)$, there exists a group R such that $\mathrm{Nor}_{\mathfrak{f}}(R) \subset f(p)$ and there exists a normal subgroup N of R with $G/C_G(\mathrm{Soc}(G)) \cong R/N$. By minimality of G, we have $G/C_G(\mathrm{Soc}(G)) \in \mathfrak{f}$. Let D be an $\mathfrak{f}$-normalizer of R. Then, DN/N is an $\mathfrak{f}$-normalizer of R/N. Since $D \in f(p)$, $DN/N \in f(p)$. Hence $G/C_G(\mathrm{Soc}(G)) \in f(p)$ and $G \in \mathrm{LF}(f) = \mathfrak{f}$, a contradiction. Thus, $C_G(\mathrm{Soc}(G)) = 1$ and G is a primitive group of type 2. Since $G \in \mathrm{LF}(f^*)$, we have $G \in f^*(p)$ for each $p \in \pi(\mathrm{Soc}(G))$. Thus, G is strongly dense with respect to $\mathfrak{f}$, a contradiction.

(b) implies (a). Assume that there exists a group $G \in b(\mathfrak{f})$ strongly dense with respect to $\mathfrak{f}$. Then, G is a monolithic primitive group. If G is of type 2, then $G = \mathrm{Lf}(f^*) = \mathfrak{f}$, a contradiction. Hence, G is a primitive group of type 1. Let p be the characteristic of $\mathrm{Soc}(G)$. Since $G \in f^*(p)$ there exists an $\mathfrak{f}$-normalizer T

of G such that $T \in f(p)$. Now, T is a complement to Soc(G) and then $G \in \mathfrak{S}_p f(p) = f(p) \subset \mathfrak{f}$, a contradiction.

(6.4) LEMMA. *Suppose that f is an S'_w-closed formation function. For every prime p, define $t(p) = (f^*(p))^{\{QR_0, S'_w\}}$. If $b(\mathfrak{f})$ is strongly wide, then $\mathfrak{f} = \mathrm{LF}(t)$.*

PROOF. It suffices to prove that $f_0 \leqq t$ by (6.2). Since $f(p)$ is an S'_w-closed formation for every prime p and f is integrated, we have $f(p) \subset f^*(p)$. By definition of $t(p)$, $f(p) \subset t(p)$. Then, $f_0(p) \subset t(p)$ and $\mathfrak{f} = \mathrm{LF}(t)$.

(6.5) THEOREM. *Assume that f and f^* are both S'_w-closed formation functions. Then $\mathfrak{f}$ has a unique maximal S'_w-closed local definition if and only if $b(\mathfrak{f})$ is strongly wide. Moreover, f^* itself is the maximal S'_w-closed local definition of $\mathfrak{f}$.*

PROOF. Suppose that $b(\mathfrak{f})$ is strongly wide. Applying the above lemma, $\mathfrak{f} = \mathrm{LF}(t)$. Since f^* is S'_w-closed, we have $t = f^*$. Further, if g is an S'_w-closed formation function such that $\mathfrak{f} = \mathrm{LF}(g)$, from (6.2), we deduce that $f_0 \leqq g \leqq f^*$. Thus f^* is the maximal S'_w-closed local definition of $\mathfrak{f}$.

Conversely, assume that g is the unique maximal S'_w-closed local definition of $\mathfrak{f}$. If $G \in b(\mathfrak{f})$ and G is strongly dense with respect to $\mathfrak{f}$, a routine argument shows that G is a primitive group of type 2. Let p be prime dividing the order of Soc(G). Define:

$$g^*(r) = \begin{cases} \{QR_0, S'_w\}(f(p) \cup \{G\}) & \text{if } r = p, \\ f(r) & \text{if } r \neq p. \end{cases}$$

It is clear that g^* is an S'_w-closed formation function. Further, if T is a group in $b(\mathfrak{f})$ which is strongly dense with respect to $\mathfrak{f}$, then T is a primitive group of type 2. Thus, there exists a prime $r \in \pi(\mathrm{Soc}(T))$ such that $r \neq p$. Then, $T \notin g^*(r) = f(r)$. On the other hand, $f(p) \cup \{G\}$ is contained in $f^*(p)$. Then, $g^*(p)$ is contained in $f^*(p)$. Thus, $f_0 \leqq g^* \leqq f^*$. By (6.2), $\mathfrak{f} = \mathrm{LF}(g^*)$. Then, $g^* \leqq g$ by maximality of g. Thus, $G \in g(p)$ and this is true for each $p \in \pi(\mathrm{Soc}(G))$. Therefore, $G \in \mathfrak{f}$, a contradiction.

In the soluble case, $b(\mathfrak{f}) \subset \mathfrak{B}_1$ and then $b(\mathfrak{f})$ is strongly wide. Therefore, we can deduce the following:

(6.6) COROLLARY (Doerk [4]). *In the universe $\mathfrak{S}$ of all finite soluble groups, every local formation possesses a unique maximal local definition.*

Finally, we give some sufficient conditions for a saturated formation of finite groups to have a maximal local definition. Recall that if $\mathfrak{X}$ is a class of groups, $h(\mathfrak{X})$ is the class of $\mathfrak{X}$-perfect groups, i.e. groups with no epimorphic images in $\mathfrak{X}$.

(6.7) LEMMA (Doerk [5]). *Let* $\mathfrak{h}$ *and* $\mathfrak{k}$ *be homomorphs and* $\mathfrak{M} = h(b(\mathfrak{k}) \cap \mathfrak{h})$. *Then* $\mathfrak{M}$ *is a largest (unique) homomorph such that* $\mathfrak{M} \cap \mathfrak{h} \subset \mathfrak{k}$.

In our case, we define for each prime p, $f^{\#}(p) = h(b(f(p)) \cap \mathfrak{f})$. By (6.7), $f^{\#}(p)$ is the largest homomorph such that $f^{\#}(p) \cap \mathfrak{f} \subset f(p)$. In fact, $f^{\#}(p) \cap \mathfrak{f} = f(p)$. Since $f^{*}(p) \cap \mathfrak{f} = f(p)$, we have that $f^{*}(p) \subset f^{\#}(p)$ for each prime p.

Suppose that for each prime $p, f^{\#}(p)$ is S'_w-closed. Then, for each prime p, $f^{\#}(p) = f^{*}(p)$. Thus, $f^{\#}$ is a formation function. Moreover, a group G is dense with respect to $\mathfrak{f}$ if and only if G is strongly dense with respect to $\mathfrak{f}$.

With similar arguments to those used in (6.5), one can prove:

(6.8) THOEREM. *Suppose that for every prime* $p, f^{\#}(p)$ *is* S'_w*-closed. Then* $\mathfrak{f}$ *possesses a unique maximal local definition if and only if* $b(\mathfrak{f})$ *is wide. In this case,* $f^{\#} = f^{*}$ *is the maximal local definition.*

Finally, using (6.1) one can easily prove:

(6.9) PROPOSITION. *In the universe* $\mathfrak{S}$ *of all finite soluble groups* $f^{\#}$ *is a formation function if and only if* $f^{\#}$ *is* S_w*-closed.*

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